

NOTE ON CLASSICAL NOTION OF LEE FORM.

PIOTR DACKO

ABSTRACT. This note is devoted to partial study of recurrent equation $d\omega = \beta \wedge \omega$, based on linear algebra of exterior forms. Such equation was considered by Lee, for non-degenerate 2-form. In this note we approach general case, when ω is arbitrary. Particularly, we extend results obtained by Lee, on odd-forms.

1. INTRODUCTION

It was noticed by Lee, [2], that equation $d\omega = \beta \wedge \omega$, for given non-degenerate exterior 2-form follows: a) $d\beta = 0$, if dimension of manifold is ≥ 6 , b) if dimension = 4, then for any 3-form κ , there is exactly 1-form β , such that $\kappa = \beta \wedge \omega$. In the original paper, there is very simple justification of the latter fact: $\kappa = \beta \wedge \omega$, is equivalent to system of linear equations, which can be resolved uniquely. Particularly, in dimension four we can find ω , with $d\beta \neq 0$. Later on Libermann, also Ślebodziński rediscovered these results, [3, 6].

In this note we try to obtain some information in general setting, when ω is arbitrary. The starting observation is that the equation $d\omega = \beta \wedge \omega$, implies $d\beta \wedge \omega = 0$. The latter can be studied point-wise, by means of some basic linear algebra.

2. PRELIMINARIES

For vector space V , $\dim V = n$, by $\Lambda^k(V^*)$ we denote the space of all k -linear totally anti-symmetric real functions (forms) on V . We set $\Lambda^0(V^*) = \mathbb{R}$ and $c \in \mathbb{R}$ is treated as constant function. Elements of $\Lambda^k(V^*)$ are called k -forms, we set

$$(1) \quad \Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \dots \oplus \Lambda^n(V^*).$$

The \wedge -product (exterior multiplication) on $\Lambda(V^*)$ is defined as usually. If $c \in \Lambda^0(V^*)$, $\omega \in \Lambda(V^*)$ then $c \wedge \omega = c\omega$. The degree $\deg \beta$, of a form is defined as a number of its arguments, by definition $\deg c = 0$, $c \in \mathbb{R}$. Interior multiplication of a $(k+1)$ -form θ and a vector $v \in V$, is a k -form $\iota_v \theta$, defined by the equation

$$(2) \quad (\iota_v \theta)(u_1, \dots, u_k) = (\deg \theta) \theta(v, u_1, \dots, u_k)$$

2000 *Mathematics Subject Classification.* 53A45, 53B99, 53C15.

Key words and phrases. Lee form, locally conformal Kaehler manifolds, almost α -cosymplectic manifolds.

where $u_1, \dots, u_k \in V$.

For arbitrary forms $\iota_x(\mu \wedge \nu) = \iota_x \mu \wedge \nu + (-1)^{\deg \mu} \mu \wedge \iota_x \nu$. For any vector $\iota_x^2 = 0$, however ι_x is exact in the sense that if $\iota_x \mu = 0$, then there is a form ν , $\deg \nu = \deg \mu + 1$, and $\iota_x \nu = \mu$. For vectors $x_1, \dots, x_k \in V$, we set $\iota_{[x_1 x_2 \dots x_k]} = \iota_{x_1} \iota_{x_2} \dots \iota_{x_k}$, changing order results $\iota_{[x_{i_1} \dots x_{i_k}]} = \pm \iota_{[x_1 \dots x_k]}$, where \pm is sign of permutation ($1 \mapsto i_1, \dots, k \mapsto i_k$). The operator $\iota_{[x_1 \dots x_j]}$, $j \geq 1$, we call j 'th-derivative and denote $\iota^{(j)}$.

With help of interior multiplication we can define pairing between k -tuples (x_1, \dots, x_k) of vectors and k -forms, by the formula

$$(3) \quad \langle [x_1 \dots x_k], \theta \rangle = \iota_{[x_1 \dots x_k]} \theta = \pm k! \theta(x_1, \dots, x_k),$$

on the right hand side \pm , is sign of the reverse ($1 \mapsto k, 2 \mapsto (k-2), \dots, k \mapsto 1$). From this definition follows that pairing is non-degenerate: if

$$\langle [x_1 \dots x_k], \theta \rangle = 0,$$

for any k -tuple, then $\theta = 0$. For $\theta = \eta_1 \wedge \dots \wedge \eta_k$ simple

$$(4) \quad \langle [x_1 \dots x_k], \theta \rangle = \pm \det |\eta_i(x_j)|, \quad i, j = 1, \dots, k.$$

If (e_1, \dots, e_n) , is an ordered base of V , and order is extended to dual forms $(\alpha_1, \dots, \alpha_n)$, $\alpha_i(e_j) = \delta_{ij}$, then

$$\iota_{[e_{i_1} \dots e_{i_k}]} \theta, \quad i_1 < \dots < i_k,$$

is coefficient of β , at term $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k}$.

For vector subspace $C \subset V$, $0 \leq p = \dim C < n$, let

$$(5) \quad C^0 = \{\alpha \in V^* \mid \alpha(x) = 0, x \in C\},$$

be a space of all 1-forms vanishing on C . For $\dim C = 0$, $C^0 = V^*$. We set $k = \dim C^0$, then $k = \text{codim } C = n - p$.

We are interested in studying properties of forms related to the pair (V, C) . By definition $(V, \{0\}) = V$. Isomorphism of (V, C) is non-degenerate linear map of V , leaving C invariant. The linear base $(e_1, \dots, e_p, \dots, e_n)$ of (V, C) is a base of V , where (e_1, \dots, e_p) , span C . By change of a base, it is understood passing from base to base of (V, C) . In similar manner we understand isomorphisms and linear bases of (V^*, C^0) . Usual duality of linear maps $f \leftrightarrow f^*$, $f : V \rightarrow V$, $f^* : V^* \rightarrow V^*$, $f^* \alpha = \beta$, $\beta(x) = \alpha(fx)$, $x \in V$, establishes duality of pairs (V, C) and (V^*, C^0) : f is isomorphism of (V, C) if and only if f^* is isomorphism of (V^*, C^0) . The same is true for the operation of base change: by duality changing bases of (V, C) is equivalent to changing bases of (V^*, C^0) . We will use these facts without explicitly referring to them.

The pair (V, C) gives rise to properly defined sub-algebra $\Lambda(C^0) \subset \Lambda(V^*)$. If $(\alpha_1, \dots, \alpha_k)$ is a base of C^0 , then $\Lambda^1(C^0) = C^0$, and $\tau \in \Lambda^l(C^0)$, $l \geq 2$, means, that τ is a sum of \wedge -products of α_i 's. Any element $\tau \in \Lambda^l(C^0)$, $l \geq 1$, nullifies C , in the sense that $\iota_{[v_1 \dots v_j]} \tau = 0$, whenever at least one of v_1, \dots, v_j is in C .

Let the $(\beta_1, \dots, \beta_{n-k})$ be such that $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k})$ is a base of V^* . A form ω can be written as a sum

$$(6) \quad \omega = \mu_{s_1} + \mu_{s_2} + \dots, \quad s_1 < s_2 < \dots,$$

where

$$(7) \quad \mu_{s_l} = \sum_{\sigma=(i_1 < \dots < i_{s_l})} \beta_{(s_l)}^\sigma \wedge \alpha_{i_1} \wedge \dots \wedge \alpha_{i_{s_l}}, \quad l = 1, 2, \dots,$$

$$(8) \quad \beta_{(s_l)}^\sigma = \sum_{j_1 < \dots < j_{m_l}} c_{\sigma, s_l}^{j_1 \dots j_{m_l}} \beta_{j_1} \wedge \dots \wedge \beta_{j_{m_l}}, \quad c_{\sigma, s_l}^{j_1 \dots j_{m_l}} = \text{const.},$$

and

$$s_l + m_l = \deg \omega, \quad l = 1, 2, \dots$$

The form μ_{s_1} will be called main part of ω and we denote it as ω^* , and by $|\omega^*|$ the common number of α_i 's in each summand of the main part, so

$$(9) \quad |\omega^*| = \deg(\alpha_{i_1} \wedge \dots \wedge \alpha_{i_{s_1}}) = s_1.$$

The difference $\omega - \omega^*$, is a *reminder*. Such decomposition depends on the choice of the base of (V^*, C^0) , however if

$$\omega = \mu_{s'_1} + \mu_{s'_2} + \dots,$$

is the respective decomposition in some other base, then $s_1 = s'_1$.

Proposition 1. *For any $(j+s)$ -form $\omega \neq 0$, $s = |\omega^*|$, $1 \leq j \leq \dim C$, there is derivative $\iota^{(j)}$, such that $\iota^{(j)}\omega \neq 0 \in \Lambda^s(C^0)$.*

Proof. We fix base of (V^*, C^0) , let $1 \leq j \leq \dim C$ be the common degree $j = \deg \beta^\sigma$, of coefficients of main part $\omega^* = \sum_\sigma \beta^\sigma \wedge \alpha_\sigma$. For vectors $v_1, \dots, v_j \in C$

$$(10) \quad \omega' = \iota_{[v_1 \dots v_j]} \omega^* = \sum_\sigma \langle [v_1 \dots v_j], \beta^\sigma \rangle \alpha_\sigma,$$

let $c^\sigma = \langle [v_1 \dots v_j], \beta^\sigma \rangle$, then for some v_1, \dots, v_j , at least one c^σ is non-zero. As forms $\alpha_\sigma \in \Lambda^s(C^0)$, in the decomposition of ω^* , are linearly independent, we have $\omega' \neq 0$, $\omega' \in \Lambda^s(C^0)$. Now it is enough to notice, that all j -derivatives $\iota_{[v_1 \dots v_j]}$, $v_1, \dots, v_j \in C$, of the reminder of ω are zero. \square

Remark 1. In the case $|\omega^*| = 0$ this result states that $\iota^{(l)}\omega = \text{const} \neq 0$. \square

3. ŚLEBODZIŃSKI LEMMA

Let $\Omega \neq 0$ be a 2-form on V , $\dim V = n + 2p \geq 3$, where $p \geq 1$, is a rank of Ω , so p is maximal integer such that $\Omega^{\wedge p} \neq 0$. The space C is now the kernel of Ω

$$C = \{x \in V \mid \iota_x \Omega = 0\},$$

then $\dim C = n$, $\dim C^0 = 2p$. Clearly $\Omega \in \Lambda(C^0)$. For $C = \{0\}$, we have $\Lambda(C^0) = \Lambda(V^*)$. We want to answer the question, under what conditions the equation

$$(11) \quad \Omega \wedge \beta = 0,$$

has non-trivial solution $\beta \neq 0$.

Lemma 1. *Let $1 \leq l \leq \dim V - 2$ and $0 \leq s \leq \min(2p, l)$, define*

$$(12) \quad \mathcal{K}_{l,s} = \left\{ \beta \in \Lambda^l(V^*) \setminus \{0\} \mid \Omega \wedge \beta = 0, s = |\beta^*| \right\},$$

then

$$(13) \quad \mathcal{K}_{l,s} = \emptyset, \quad 0 \leq s < \min(p, l),$$

$$(14) \quad \mathcal{K}_{l,s} \neq \emptyset, \quad s \geq p.$$

Proof. Let $\beta \neq 0$, be a l -form, $l \geq 1$, $s = |\beta^*|$, and define β' as follows:

$$(15) \quad \beta' = \begin{cases} \beta, & \text{if } \beta \in \Lambda^l(C^0), \\ \iota^{(j)}\beta \in \Lambda^s(C^0), & \iota^{(j)}\beta \text{ is as in the Proposition 1,} \end{cases}$$

here $\iota^{(j)} = \iota_{[v_1 \dots v_j]}$, for some vectors $v_1, \dots, v_j \in C$. For $\beta \neq \beta'$

$$(16) \quad 0 = \iota_{[v_1 \dots v_j]}(\Omega \wedge \beta) = \Omega \wedge (\iota_{[v_1 \dots v_j]}\beta) = \Omega \wedge \beta'.$$

Thus, in any case we have

$$(17) \quad \Omega \wedge \beta = 0 \quad \Rightarrow \quad \Omega \wedge \beta' = 0.$$

The right hand side of this implication follows, that β' can not be a constant, thus $s > 0$. By induction

$$(18) \quad \Omega \wedge \beta' = 0 \quad \Rightarrow \quad \Omega^{\wedge(i+1)} \wedge \iota_{[v_1 \dots v_i]}\beta' = 0,$$

for any vectors $v_1, \dots, v_i \in V$. For $s < p$

$$(19) \quad \Omega \wedge \beta' = 0 \quad \Rightarrow \quad \langle [v_1 \dots v_s], \beta' \rangle \Omega^{s+1} = 0,$$

the last equation follows $\langle [v_1 \dots v_s], \beta' \rangle = 0$. Non-degeneracy of $\langle \cdot, \cdot \rangle$, implies $\beta' = 0$. Hence $s \geq p$.

Simple dimension considerations, $\dim \Lambda^s(C^0) = \binom{2p}{s}$, follow, that for $s \geq p$ there is $\beta' \neq 0 \in \Lambda^s(C^0)$, such that $\Omega \wedge \beta' = 0$. Now we may take $\beta = \tau \wedge \beta' \neq 0 \in \mathcal{K}_{l,s}$, for some $(l-k)$ -form τ , then $\Omega \wedge \beta = 0$. \square

Corollary 1. *The immediate consequence of the above result is that degree of $\beta \neq 0$, in (11) is always bounded below by the rank of ω , $\deg \beta \geq p$.*

Remark 2. Let $\omega_1 \wedge \omega_2 = 0$, for 2-forms ω_1, ω_2 , on a vector space V^n . If $\omega_1 \neq 0$ and $\omega_2 \neq 0$, then ω_1, ω_2 are at most of rank 2; there are 1-forms $(\eta_1, \eta_2, \beta_1, \beta_2)$, such that

$$(20) \quad \omega_1 = \eta_1 \wedge \eta_2 + c \beta_1 \wedge \beta_2, \quad c \in \mathbb{R},$$

and we have the following five possibilities for ω_2

$$(21) \quad \eta_1 \wedge \eta_2 - c\beta_1 \wedge \beta_2, \quad b\eta_i \wedge \beta_j, \quad b \neq 0 \in \mathbb{R}, \quad i, j = 1, 2,$$

□

The operation $\beta \mapsto \omega \wedge \beta$ defines family of maps $\lambda^k : \Lambda^k(V) \rightarrow \Lambda^{k+2}(V)$,

$$(22) \quad \Lambda^k(V) \ni \beta \mapsto \omega \wedge \beta \in \Lambda^{k+2}(V), \quad k = 0, \dots, n,$$

clearly $\lambda^{n-1} = \lambda^n = 0$.

Corollary 2. *Let ω has trivial kernel, so $\dim V = 2p$, $\omega^{\wedge p} \neq 0$. Then maps λ^k are 1-1, for $k \leq p-1$. In particular λ^{p-1} is isomorphism of the spaces $\Lambda^{p-1}(V)$ and $\Lambda^{p+1}(V)$.*

For $p = 2$, this Corollary rediscovers the result of Lee, [2] (cf. Ślebodziński, [6], p. 314)

Remark 3. It is unknown to the author, but it is expected that for $k \geq p$, λ^k is epimorphism. Moreover, it is possible in combinatorial way, to describe forms spanning the kernel of λ^k . □

4. MAIN RESULT AND ITS APPLICATIONS

At the beginning let consider two examples.

Example 1. Let $d\beta = \eta \wedge \beta$, for non-zero 1-forms β and η . Notice, that η in this equation is non-unique: two solutions differs by $f\beta$, for a function f . However, $\beta \wedge d\beta = 0$, wich means, that the kernel of β is involutive, and locally $\beta = f\beta_0$, for some closed 1-form β_0 . Then $d\beta = d \ln |f| \wedge \beta$, so we may set $\eta = d \ln |f|$. In conclusion we obtain, that the system

$$(23) \quad d\beta = \eta \wedge \beta, \quad d\eta = 0,$$

determines η uniquely. □

Example 2. Let ω be a 2-form of maximal rank on a manifold \mathcal{M} . In other words the kernel of ω is trivial or one-dimensional. Let again, $d\omega = \eta \wedge \omega$. Here η is unique if $\dim \mathcal{M} \geq 4$. No additional assumptions are needed. □

On a manifold the rank of a 2-form may vary from point to point. For a 2-form ω we denote by $r(\omega)$ function which associates to each point the rank of ω at this point. Let $r(\omega) = k$, at a point. Then $\omega^k \neq 0$ and $r(\omega) \geq k$ on some neighborhood \mathcal{U} of this point. This argument proves, that $r(\omega)$ is lower semi-continuous on any manifold.

Theorem 1. *Let ω be smooth 2-form on connected, smooth manifold \mathcal{M} , $\dim \mathcal{M} \geq 4$, such that*

$$(24) \quad d\omega = \beta \wedge \omega,$$

for some 1-form β . It is assumed that the set of points, where $\omega = 0$ is nowhere dense in \mathcal{M} . Let define

$$(25) \quad \mathcal{A} = \{r(\omega) > 2\}, \quad \mathcal{B} = \{d\beta \neq 0, \omega \neq 0\}, \quad \mathcal{C} = \{r(\omega) \leq 1\},$$

then

- a) \mathcal{A} is open; if non-empty, then $d\beta = 0$ on \mathcal{A} ;
- b) If \mathcal{B} is non-empty, then $1 \leq r(d\beta)$, $r(\omega) \leq 2$ on \mathcal{B} ;
- c) For \mathcal{C} nowhere dense β is unique.

Particularly, it follows from a) and b), that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Proof. At first we notice that $d\omega = \beta \wedge \omega$ follows $d\beta \wedge \omega = 0$, at any point of \mathcal{M} .

\Rightarrow a) \mathcal{A} is open comes from the fact, that $r(\omega)$ is lower semi-continuous. Let $\mathcal{A} \neq \emptyset$. Take $p \in \mathcal{A}$. Then from $(d\beta)_p \wedge \omega_p = 0$ and Lemma 1, cf. also Corollary 1, follow $(d\beta)_p = 0$.

\Rightarrow b) In the case $\mathcal{B} \neq \emptyset$, let $p \in \mathcal{B}$, for $(d\beta)_p \neq 0$ and $(d\beta)_p \wedge \omega_p = 0$, by the Remark 3.1, the ranks satisfy $1 \leq r(d\beta)$, $r(\omega) \leq 2$ at p .

\Rightarrow c) By assumption set of points where $r(\omega) \geq 2$, is open and dense; at each point of this set β is unique, hence is unique everywhere. \square

For ω of maximal rank and $\dim \mathcal{M} \geq 6$, we have $\mathcal{A} = \mathcal{M}$.

Corollary 3. *Let $\dim \mathcal{M} \geq 6$, and ω be a 2-form of maximal rank, such that $d\omega = \beta \wedge \omega$. Then β is closed, $d\beta = 0$.*

In particular case of even-dimensional manifolds we can restate the following result

Corollary 4 (Lee-Liebermann-Ślebodziński, [2, 3, 6]). *Let ω be a non-degenerate 2-form on even-dimensional manifold \mathcal{M} . Assume that $d\omega = \beta \wedge \omega$. If $\dim \mathcal{M} \geq 6$, then β is closed, $d\beta = 0$.*

We emphasize that the Corollary 3, is an enhancement of the above mentioned result, for it holds also for odd-dimensional manifolds.

Remark 4. In general, let (\mathcal{M}, J, g) , be an almost Hermitian manifold, where ω is now the fundamental form of \mathcal{M} , $\omega(X, Y) = g(JX, Y)$, and

$$(26) \quad d\omega = \beta \wedge \omega, \quad d\beta = 0,$$

so, β is closed. If we focus only on a sufficiently small open disk $\mathcal{D} \subset \mathcal{M}$, then $\beta_{\mathcal{D}} = \beta|_{\mathcal{D}}$ is exact on \mathcal{D} , $f : \mathcal{D} \rightarrow \mathbb{R}$, $df = \beta_{\mathcal{D}}$. Now, the structure $(J|_{\mathcal{D}}, e^{-f}g|_{\mathcal{D}})$ is an almost Kähler structure on \mathcal{D} . According to Vaisman [7], such manifolds are called locally conformal (almost) Kähler, (l.c.a.K. manifolds). The form β is called Lee form.

In modern literature, many authors, when referring the notion of l.c.a.K manifolds, are using (26). Of course such definition is correct but redundant and, worse, can be confusing, suggesting that $d\beta = 0$ is general requirement. If $\dim \mathcal{M} \geq 6$, once the fundamental form satisfies $d\omega = \beta \wedge \omega$, the Lee form is automatically closed, $d\beta = 0$. \square

Remark 5. An almost contact metric manifold $(\mathcal{M}, \phi, \xi, \eta, g)$ is called almost α -cosymplectic [5], if

$$(27) \quad d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

where Φ is fundamental form of \mathcal{M} , $\Phi(X, Y) = g(\phi X, Y)$ and α is a function on \mathcal{M} . Form Φ is non-degenerate and the Reeb vector field ξ , spans kernel of Φ , $\iota_\xi \Phi = 0$. Particularly, for $\alpha = 1$, (27) defines class of almost Kenmotsu manifolds, [1].

If $\dim \mathcal{M} > 5$, then the condition $d\Phi = 2\alpha\eta \wedge \Phi$, yields, that $\alpha\eta$, is closed

$$(28) \quad d(\alpha\eta) = d\alpha \wedge \eta = 0.$$

Hence $d\alpha = f\eta$, $f = \xi\alpha$. There are two simple remarks:

- a) there is no need to require $d\eta = 0$, in the definition of almost Kenmotsu manifolds, for dimensions > 5 ,
- b) let drop the assumption $d\eta = 0$, in (27), nevertheless, near points where $d\alpha \neq 0$, we have $d\eta \wedge \eta = 0$, so the kernel distribution $\eta = 0$, on such domains is completely integrable.

□

Example 3. Let

$$(29) \quad \omega_f = e^f dx^1 \wedge dx^2 + dy^1 \wedge dy^2,$$

be defined on \mathbb{R}^4 , $v = (x^1, x^2, y^1, y^2) \in \mathbb{R}^4$. Then $d\omega_f = \beta_f \wedge \omega_f$, and for particular choices of the function f , the form β satisfies a priori imposed conditions. Set $f = f_0 = x^1 y^1 + x^2 y^2$, $\omega_0 = \omega_{f_0}$, then

$$(30) \quad \beta_0 = x^1 dy^1 + x^2 dy^2, \quad d\beta_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2,$$

clearly $d\beta_0 \wedge \omega_0 = 0$.

□

Example 4. On the basis of the previous example, let construct on a manifold $\mathcal{M} = \mathbb{R}_+ \times \mathbb{R}^4$, $p = (t, v) \in \mathcal{M}$, $t > 0$, $v \in \mathbb{R}^4$, a structure consisting of a pair (η, Φ) , where η is a 1-form, $d\eta = 0$, Φ is a 2-form, and $\eta \wedge \Phi^{\wedge 2}$ is a volume (oriented) on \mathcal{M} . Moreover

$$(31) \quad d\Phi = \gamma \wedge \Phi, \quad \text{and } \gamma \text{ is a contact form on } \mathcal{M}.$$

Directly, we verify that forms $\eta = dt$, $\Phi = t\omega_0$, satisfy the required conditions,

$$(32) \quad d\Phi = \gamma \wedge \Phi, \quad \gamma = d \ln t + \beta_0 = d \ln t + x^1 dy^1 + x^2 dy^2,$$

now it is clear, that γ is contact form.

□

In [4], there is given example of four-dimensional Lie group with ω left-invariant, non-degenerate, and $d\omega = \beta \wedge \omega$, $d\beta \neq 0$.

REFERENCES

- [1] G. Dileo, A.M. Pastore, *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 343–354.
- [2] H. C. Lee *A kind of even dimensional differential geometry and its applications to exterior calculus*, Amer. J. Math. **65** (1943), 433–438.
- [3] P. Libermann, *Sur le probleme d'equivalence de certaines structures infinitesimales*, Ann. Mat. Pura Appl. **36** (1954), 27–120.
- [4] Z. Olszak, *Four-dimensional parahermitian manifolds*, Tensor N.S. **56** (1995), 215–226.

- [5] H.Öztürk, N. Aktan, C. Murathan, *Almost α -cosymplectic (κ, μ, ν) -spaces*, Submitted. Available in arXiv:1007.0527 [math. DG].
- [6] W. Ślebodziński, *Exterior forms and their applications*, PWN – Polish Scientific Publishers, Warszawa 1970.
- [7] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 338–351.

E-mail address: piotrdacko@yahoo.com